

# LIE $n$ -RACKS

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## Abstract

In this paper, we introduce the category of Lie  $n$ -racks and generalize several results known on racks. In particular, we show that the tangent space of a Lie  $n$ -Rack at the neutral element has a Leibniz  $n$ -algebra structure. We also define a cohomology theory of  $n$ -racks.

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## 1 Introduction and Generalities

Manifold with  $n$ -ary operations on their function algebra have been an important area of attraction for many years. It all started with Nambu [16] introducing in 1973, an  $n$ -ary generalization of Hamiltonian Dynamics by the  $n$ -ary Poisson bracket, also known as the Nambu bracket. Since then, Nambu-Poisson structures have been seriously studied. The work of Takhtajin [18] in 1994 on the foundation of the theory of Nambu-Poisson manifolds, and recent applications of Nambu brackets to brane theory [2] are just a few to mention. In particular, Nambu algebras are infinite dimensional Filippov algebras. Also known in literature as  $n$ -Lie algebras (not to mistake with Lie  $n$ -algebras), Filippov algebras were first introduced in 1895 by Filippov [9] and was generalized to the concept of Leibniz  $n$ -algebras by Casas, Loday and Pirashvili [5]. Both concepts are important in Nambu Mechanics [16]. Applications of Filippov algebras are found in String theory [2] and in Yang-Mills theory [10].

One of the most important problems in Leibniz algebra theory is the coquecigrué problem (a generalization of Lie's third theorem to Leibniz algebras) which consists of

finding a generalization of groups whose tangent algebra structure corresponds to a Leibniz algebra. Loday dubbed these objects “coquecigrues” [15] as no properties were foreseen on them. While attempting to solve this problem, Kinyon [13] showed that the tangent space at the neutral element of a Lie rack has a Leibniz algebra structure. In his doctoral thesis, Covez [6] provided a local answer to the coquecigrue problem by showing that every Leibniz algebra becomes integrated into a local augmented Lie rack. Thus Lie racks seem to be the best candidates for “coquecigrues”.

Meanwhile, Grabowski and Marmo [11] provided in the same order of idea an important connection between Filippov algebras and Nambu- Lie Groups. All these ideas suggest a new mathematical structure by extending the binary operation of Lie racks to an  $n$ -ary operation. This yields the introduction of Lie  $n$ -racks and generalizes Lie racks from the case  $n = 2$ . It turns out that one can extend Kinyon’s result to Leibniz algebras via Lie  $n$ -racks. If it is reasonable to talk about the coquecigrue problem for Leibniz  $n$ -algebras, this work certainly opens a lead towards a solution.

In [8], Fenn, Rourke and sanderson introduced a cohomology theory for racks which was modified in [4] by Carter, Jelsovsky, Kamada, Landford and Saito to obtain quandle cohomology, and several results have been recently established. We use these cohomology theories in section 4 to define cohomology theories on  $n$ -racks and  $n$ -quandles.

let us recall a few definitions. Given a field  $\mathfrak{K}$  of characteristic different to 2, a Leibniz  $n$ -algebra [5] is defined as a  $\mathfrak{K}$ -vector space  $\mathfrak{g}$  equipped with an  $n$ -linear operation  $[-, \dots, -] : \mathfrak{g}^{\otimes n} \longrightarrow \mathfrak{g}$  satisfying the identity

$$[x_1, \dots, x_{n-1}, [y_1, y_2, \dots, y_n]] = \sum_{i=1}^n [y_1, \dots, y_{i-1}, [x_1, \dots, x_{n-1}, y_i], y_{i+1}, \dots, y_n] \quad (1.1)$$

Notice that in the case where the  $n$ -ary operation  $[-, \dots, -]$  is antisymmetric in each pair of variables, i.e.,

$$[x_1, x_2, \dots, x_i, \dots, x_j, \dots, x_n] = -[x_1, x_2, \dots, x_j, \dots, x_i, \dots, x_n]$$

or equivalently  $[x_1, x_2, \dots, x, \dots, x, \dots, x_n] = 0$  for all  $x \in G$ , the Leibniz  $n$ -algebra becomes a Filippov algebra (more precisely a  $n$ -Filippov algebra). Also, a Leibniz 2-algebra is exactly a Leibniz algebra [14, p.326] and become a Lie algebra if the binary operation  $[\ , \ ]$  is skew symmetric.

If  $\mathfrak{g}$  is a vector space endowed with an  $n$ -linear operation  $\sigma : \mathfrak{g} \times \mathfrak{g} \times \dots \times \mathfrak{g} \longrightarrow \mathfrak{g}$ , then a map  $D : \mathfrak{g} \longrightarrow \mathfrak{g}$  is called a derivation with respect to  $\sigma$  if

$$D(\sigma(x_1, \dots, x_n)) = \sum_{i=1}^n \sigma(x_1, \dots, D(x_i), \dots, x_n).$$

A Nambu-Lie group  $(G, P)$  is a Lie group  $G$  with a rank  $n$   $G$ -multiplicative Nambu tensor  $P$  i.e.,

$$P(g_1, g_2) = L_{g_1^*} P(g_2) + R_{g_2^*} P(g_1),$$

where  $L_{g_1^*}$  and  $R_{g_2^*}$  denote respectively left and right translations in  $G$ . More on Nambu manifolds can be found in [19]

The following proposition provides an important connection between Nambu-Lie groups and Filippov algebras. It is mainly stated in this paper because it shows an analogy between the connection of filippov algebras with Nambu-Lie groups, and the connection of Leibniz  $n$ -algebras with Lie  $n$ -racks provided in section 4.

**Proposition 1.1.** (*[11]*) *Let  $(G, P)$  be a Nambu-Lie group and let  $\delta_p : \mathfrak{g} \longrightarrow \wedge^n(\mathfrak{g})$  denote the intrinsic derivative  $\delta_P(X) = L_X(P)(e)$  where  $L_X$  denotes the Lie derivative and  $e$  the identity element of  $G$ . Then the map  $\delta_P^* : \wedge^n(\mathfrak{g}^*) \longrightarrow \mathfrak{g}^*$  defines a filippov bracket on  $\mathfrak{g}^*$ .*

A lie Rack  $(R, \circ, 1)$  is a smooth manifold  $R$  with a binary operation  $\circ$  and a specific element  $1 \in R$  such that the following conditions are satisfied:

- $x \circ (y \circ z) = (x \circ y) \circ (x \circ z)$
- for each  $x, y \in R$ , there exists a unique  $a \in R$  such that  $x \circ a = y$
- $1 \circ x = x$  and  $x \circ 1 = 1$  for all  $x \in R$
- the operation  $\circ : R \times R \longrightarrow R$  is a smooth mapping.

## 2 $n$ -racks

**Definition 2.1.** : *A left  $n$ -rack<sup>1</sup> (right  $n$ -racks are defined similarly)  $(R, [-, \dots, -]_R)$  is a set  $R$  endowed with an  $n$ -ary operation  $[-, \dots, -]_R : R \times R \times \dots \times R \longrightarrow R$  such that*

$$1. [x_1, \dots, x_{n-1}, [y_1, \dots, y_{n-1}]_R]_R = [[x_1, \dots, x_{n-1}, y_1]_R, \dots, [x_1, \dots, x_{n-1}, y_n]_R]_R \quad (2.1)$$

*(This is the left distributive property of  $n$ -racks)*

$$2. \text{ For } a_1, \dots, a_{n-1}, b \in R, \text{ there exists a unique } x \in R \text{ such that } [a_1, \dots, a_{n-1}, x]_R = b.$$

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<sup>1</sup>2-racks coincide with Racks. They were introduced in 1959 by G. Wraith and J. Conway[3], and have been used in various topics of mathematics. In particular, they are mainly used in topology to provide invariants for knots [12].

If in addition there is a distinguish element  $1 \in R$ , such that

$$3. [1, \dots, 1, y]_R = y \quad \text{and} \quad [x_1 \dots, x_{n-1}, 1]_R = 1 \quad \text{for all } x_1, \dots, x_{n-1} \in R,$$

then  $(R, [\dots]_R, 1)$  is said to be a pointed  $n$ -rack.

A  $n$ -rack is a weak  $n$ -quandle if it further satisfies

$$[x, x, \dots, x, x]_R = x \quad \text{for all } x \in R.$$

A  $n$ -rack is a  $n$ -quandle if it further satisfies

$$[x_1, x_2, \dots, x_{n-1}, y]_R = y \quad \text{if } x_i = y \quad \text{for some } i \in \{1, 2, \dots, n-1\}.$$

A  $n$ -quandle (resp weak  $n$ -quandle) is a  $n$ -kei (resp weak  $n$ -kei) if it further satisfies

$$[x_1, \dots, x_{n-1}, [x_1, \dots, x_{n-1}, y]] = y \quad \text{for all } x_1, \dots, x_{n-1}, y \in R.$$

For  $n = 2$ , one recovers racks, quandles[12] and keis [17]. Note also that  $n$ -quandles are also weak  $n$ -quandles, but the converse is not true for  $n > 2$ ; See example 2.3.

**Definition 2.2.** Let  $R, R'$  be  $n$ -racks. A function  $\alpha : R \longrightarrow R'$  is said to be a homomorphism of  $n$ -racks if

$$\alpha([x_1, \dots, x_n]_R) = [\alpha(x_1), \alpha(x_2), \dots, \alpha(x_n)]_{R'} \quad \text{for all } x_1, x_2, \dots, x_n \in R.$$

We may thus form the category  ${}_n\text{pRACK}$  of pointed  $n$ -racks and pointed  $n$ -rack homomorphisms.

**Example 2.3.** A  $\mathbf{Z}_4$ -module  $M$  endowed with the operation  $[-, \dots, -]_M$  defined by

$$[x_1, \dots, x_n]_M = 2x_1 + 2x_2 + \dots + 2x_{n-1} + x_n$$

is a  $n$ -rack that is a weak  $n$ -kei if  $n$  is odd. Indeed

$$\begin{aligned} & [[x_1, \dots, x_{n-1}, y_1]_M, \dots, [x_1 \dots, x_{n-1}, y_n]_M, ]_M = \\ &= \left( \sum_{i=1}^{n-1} 2(2x_1 + 2x_2 + \dots + 2x_{n-1} + y_i) \right) + (2x_1 + 2x_2 + \dots + 2x_{n-1} + y_n) \\ &= \left( \sum_{i=1}^{n-1} 2x_i + 2y_i \right) + y_n \\ &= [x_1, \dots, x_{n-1}, [y_1, \dots, y_n]_M]_M. \end{aligned}$$

Therefore (2.1) is satisfied. One easily checks the other axioms.

**Example 2.4.** Let  $\Gamma := \mathbf{Z}[t^{\pm 1}, s]/(s^2 + ts - s)$ . Any  $\Gamma$ -module  $M$  endowed with the operation  $[-, \dots, -]_M$  defined by

$$[x_1, \dots, x_n]_M = sx_1 + sx_2 + \dots + sx_{n-1} + tx_n$$

is a  $n$ -rack that generalizes the Alexander quandle when  $s = 1 - t$ . Indeed

$$\begin{aligned} & [[x_1, \dots, x_{n-1}, y_1]_M, \dots, [x_1, \dots, x_{n-1}, y_n]_M, ]_M = \\ &= \left( \sum_{i=1}^{n-1} s(sx_1 + sx_2 + \dots + sx_{n-1} + ty_i) \right) + t(sx_1 + sx_2 + \dots + sx_{n-1} + ty_n) \\ &= (s^2 + st) \left( \sum_{i=1}^{n-1} x_i \right) + ts \left( \sum_{i=1}^{n-1} y_i \right) + t^2 y_n \\ &= [x_1, \dots, x_{n-1}, [y_1, \dots, y_n]_M]_M \quad \text{since } s^2 + st = s. \end{aligned}$$

Therefore (2.1) is satisfied. One easily checks the axiom 2.

Note that for  $t = 1$  and  $s = 2$ , this coincides with example 2.3.

**Example 2.5.** A group  $G$  endowed with the operation  $[-, \dots, -]_G$  defined by

$$[x_1, \dots, x_n]_G = x_1 x_2 \dots x_{n-1} x_n x_{n-1}^{-1} x_{n-2}^{-1} \dots x_1^{-1},$$

is a pointed weak  $n$ -quandle (pointed by  $1 \in G$ ). Indeed,

$$\begin{aligned} [x_1, \dots, x_{n-1}, [y_1, \dots, y_n]_G]_G &= x_1 x_2 \dots x_{n-1} [y_1, \dots, y_n]_G x_{n-1}^{-1} x_{n-2}^{-1} \dots x_1^{-1} \\ &= x_1 x_2 \dots x_{n-1} y_1 y_2 \dots y_{n-1} y_n y_{n-1}^{-1} y_{n-2}^{-1} \dots y_1^{-1} x_{n-1}^{-1} x_{n-2}^{-1} \dots x_1^{-1}. \end{aligned}$$

on the other hand,

$$\begin{aligned} & [[x_1, \dots, x_{n-1}, y_1]_G, \dots, [x_1, \dots, x_{n-1}, y_n]_G, ]_G = \\ &= [x_1 x_2 \dots x_{n-1} y_1 x_{n-1}^{-1} x_{n-2}^{-1} \dots x_1^{-1}, \dots, x_1 x_2 \dots x_{n-1} y_n x_{n-1}^{-1} x_{n-2}^{-1} \dots x_1^{-1}]_G \\ &= x_1 x_2 \dots x_{n-1} y_1 y_2 \dots y_{n-1} y_n y_{n-1}^{-1} y_{n-2}^{-1} \dots y_1^{-1} x_{n-1}^{-1} x_{n-2}^{-1} \dots x_1^{-1} \end{aligned}$$

by cancellation. Therefore (2.1) is satisfied. One easily checks the other axioms.

This determines a functor  $\mathfrak{F} : GROUP \longrightarrow {}_n pRACK$  from the category of groups to the category of pointed  $n$ -racks. The functor  $\mathfrak{F}$  is faithful and has a left adjoint  $\mathfrak{F}'$  defined as follows: Given a pointed  $n$ -rack  $R$ , one constructs a Group

$$G_R = \langle R \rangle / I$$

where  $\langle R \rangle$  stands for the free group on the elements of  $R$  and  $I$  is the normal subgroup generated by the set

$$\{(x_1^{-1}x_2^{-1} \dots x_{n-1}^{-1}x_n^{-1}x_{n-1}x_{n-2} \dots x_1)([x_1, \dots, x_n]_R) \text{ with } x_i \in R, i = 1, 2, \dots, n\}.$$

That  $\mathfrak{F}'$  is left adjoint to  $\mathfrak{F}$  is a consequence of the following proposition which extends to  $n$ -racks a well-known result on the category of racks.

**Proposition 2.6.** *Let  $G$  be a group and let  $R$  be a  $n$ -rack. For a morphism of  $n$ -racks  $\alpha : R \longrightarrow \mathfrak{F}(G)$ , there is a unique morphism of groups  $\alpha_* : \mathfrak{F}'(R) \longrightarrow G$  such that the following diagram commute.*

$$\begin{array}{ccc} \mathfrak{F}'(R) & \xrightarrow{\alpha_*} & G \\ \uparrow & & \uparrow id \\ R & \xrightarrow{\alpha} & \mathfrak{F}(G) \end{array}$$

*Proof.* By the universal property of free groups, there is a unique morphism of groups  $\beta : \langle R \rangle \longrightarrow G$  such that  $\alpha = \beta|_R$ . In particular, for all  $x_i \in R, i = 1, 2, \dots, n$

$$\begin{aligned} \beta((x_1^{-1}x_2^{-1} \dots x_{n-1}^{-1}x_n^{-1}x_{n-1}x_{n-2} \dots x_1)([x_1, \dots, x_n]_R)) &= \\ &= \alpha((x_1^{-1}x_2^{-1} \dots x_{n-1}^{-1}x_n^{-1}x_{n-1}x_{n-2} \dots x_1)([x_1, \dots, x_n]_R)) = 1. \end{aligned}$$

The result follows by the universal property of quotient groups. □

**Example 2.7.** *Any rack  $(R, \circ, 1)$  is also a  $n$ -rack under the  $n$ -ary operation defined by*

$$[x_1, x_2, \dots, x_n]_R = x_1 \circ (x_2 \circ (\dots (x_{n-1} \circ x_n) \dots)).$$

*This process determines a functor  $\mathfrak{G} : pRACK \longrightarrow {}_n pRACK$ , which has as left adjoint, the functor  $\mathfrak{G}' : {}_n pRACK \longrightarrow pRACK$  defined as follows: Given a pointed  $n$ -rack  $(R, [- \dots -], 1)$ , then  $R^{\times(n-1)}$  endowed with the binary operation*

$$(x_1, x_2, \dots, x_{n-1}) \circ (y_1, y_2, \dots, y_{n-1}) = ([x_1, \dots, x_{n-1}, y_1]_R, \dots, [x_1, \dots, x_{n-1}, y_{n-1}]_R) \quad (2.2)$$

is a rack pointed at  $(1, 1, \dots, 1)$ . Let us observe that if  $R$  is a  $n$ -quandle, then  $R^{\times(n-1)}$  is a quandle.

**Definition 2.8.** Let  $R$  be a pointed  $n$ -rack and let  $S_R = \{f : R \longrightarrow R, f \text{ is a bijection}\}$ . Then define  $\phi : R \times R \times \dots \times R \longrightarrow {}_n\text{Aut}(R)$  by

$$\phi(x_1, \dots, x_{n-1})(y) = [x_1 \dots, x_{n-1}, y]_R \quad \text{for all } y \in R$$

where

$${}_n\text{Aut}(R) = \{\xi \in S_R / \xi([x_1, \dots, x_n]_R) = [\xi(x_1), \dots, \xi(x_n)]_R\}.$$

That  $\phi$  is well-defined is a direct consequence of the axiom 2 of definition 2.1.

**Proposition 2.9.** Let  $(R, [-, \dots, -]_R, 1)$  is a  $n$ -rack, then for all  $x_1, \dots, x_{n-1} \in R$ ,  $\phi(x_1, \dots, x_{n-1})$  operates on  $R$  by  $n$ -rack automorphism, i.e.  $\phi(x_1, \dots, x_{n-1}) \in {}_n\text{Aut}(R)$ .

*Proof.* :

$$\begin{aligned} \phi(x_1, \dots, x_{n-1})([y_1, \dots, y_n]_R) &= [x_1, \dots, x_{n-1}, [y_1, \dots, y_n]_R]_R \\ &= [[x_1, \dots, x_{n-1}, y_1]_R, \dots, [x_1, \dots, x_{n-1}, y_n]_R]_R \text{ by (2.1)} \\ &= [\phi(x_1, \dots, x_{n-1})(y_1), \dots, \phi(x_1, \dots, x_{n-1})(y_n)]_R \end{aligned}$$

□

### 3 A (co)homology theory on $n$ -Racks

Recall that for a rack  $(X, \circ)$ , one defines [4] the rack homology  $H_*^R(X)$  of  $X$  as the homology of the chain complex  $\{C_k^R(X), \partial_k\}$  where  $C_k^R(X)$  is the free abelian group generated by  $k$ -uples  $(x_1, x_2, \dots, x_k)$  of elements of  $X$  and the boundary maps  $\partial_k : C_k^R(X) \longrightarrow C_{k-1}^R(X)$  are defined by

$$\begin{aligned} \partial_k(x_1, x_2, \dots, x_k) &= \sum_{i=2}^k (-1)^i ((x_1, x_2, \dots, x_{i-1}, x_{i+1}, \dots, x_k) \\ &\quad - (x_1 \circ x_i, x_2 \circ x_i, \dots, x_{i-1} \circ x_i, x_{i+1}, \dots, x_k)) \end{aligned}$$

for  $k \geq 2$  and  $\partial_k = 0$  for  $k \leq 1$ . If  $X$  is a quandle, the subgroups  $C_k^D(X)$  of  $C_k^R(X)$  generated by  $k$ -tuples  $(x_1, x_2, \dots, x_k)$  with  $x_i = x_{i+1}$  for some  $i$ ,  $1 \leq i < k$  form a subcomplex  $C_*^D(X)$  of  $C_*^R(X)$  whose homology  $H_*^D(X)$  is called the degeneration homology of  $X$ . The homology  $H_*^Q(X)$  of the quotient complex  $\{C_k^Q(X) = C_k^R(X)/C_k^D(X), \partial_k\}$  is called the quandle homology of  $X$ .

Now let  $\mathcal{X}$  be a  $n$ -rack. We showed in example 2.7 that  $\mathcal{X}^{\times(n-1)}$  endowed with the binary operation defined by (2.2) is a rack.  $\mathcal{X}^{\times(n-1)}$  is a quandle if  $\mathcal{X}$  is a  $n$ -quandle.

**Definition 3.1.** We define the chain complexes  ${}_nC_*^R(\mathcal{X}) := C_*^R(\mathcal{X}^{\times(n-1)})$  if  $\mathcal{X}$  is a  $n$ -rack,  ${}_nC_*^D(\mathcal{X}) := C_*^D(\mathcal{X}^{\times(n-1)})$  and  ${}_nC_*^Q(\mathcal{X}) := C_*^Q(\mathcal{X}^{\times(n-1)})$  if  $\mathcal{X}$  is a  $n$ -quandle.

**Definition 3.2.** Let  $\mathcal{X}$  be a  $n$ -rack. The  $k$ th  $n$ -rack homology group of  $\mathcal{X}$  with trivial coefficient is defined by

$$H_k^R(\mathcal{X}) = H_k({}_nC_*^R(\mathcal{X})).$$

**Definition 3.3.** Let  $\mathcal{X}$  be a  $n$ -quandle.

1. The  $k$ th  $n$ -degeneration homology group of  $\mathcal{X}$  with trivial coefficient is defined by

$$H_k^D(\mathcal{X}) = H_k({}_nC_*^D(\mathcal{X})).$$

2. The  $k$ th  $n$ -quandle homology group of  $\mathcal{X}$  with trivial coefficient is defined by

$$H_k^Q(\mathcal{X}) = H_k({}_nC_*^Q(\mathcal{X})).$$

**Definition 3.4.** Let  $A$  be an abelian group, we define the chain complexes

$${}_nC_*^W(\mathcal{X}; A) = {}_nC_*^W(\mathcal{X}) \otimes A, \quad \partial = \partial \otimes id \quad \text{with } W = D, R, Q.$$

1. The  $k$ th  $n$ -rack homology group of  $\mathcal{X}$  with coefficient in  $A$  is defined by

$$H_k^R(\mathcal{X}; A) = H_k({}_nC_*^R(\mathcal{X}; A)).$$

2. The  $k$ th  $n$ -degenerate homology group of  $\mathcal{X}$  with coefficient in  $A$  is defined by

$$H_k^D(\mathcal{X}; A) = H_k({}_nC_*^D(\mathcal{X}; A)).$$

3. The  $k$ th  $n$ -quandle homology group of  $\mathcal{X}$  with coefficient in  $A$  is defined by

$$H_k^Q(\mathcal{X}; A) = H_k({}_nC_*^Q(\mathcal{X}; A)).$$

One defines the cohomology theory of  $n$ -racks and  $n$ -quandles by duality. Note that for  $n = 2$ , one recovers the homology and cohomology theories defined by Carter, Jelsovsky, Kamada, Landford and Saito [4].

**Remark 3.5.** Since  $\mathcal{X}^{\times(n-1)}$  carries most of the properties of  $\mathcal{X}$ , several results established on racks are valid on  $n$ -racks. For instance; if  $\mathcal{X}$  is finite, then  $\mathcal{X}^{\times(n-1)}$  is also finite. Cohomology of finite racks were studied by Etingof and Graña in [7]



## 4 From Lie $n$ -racks to Leibniz $n$ -algebras

In this section we define the notion of Lie  $n$ -racks and provide a connection with Leibniz  $n$ -algebras. Throughout the section,  $T_1$  denotes the tangent functor.

**Definition 4.1.** A Lie  $n$ -rack  $(R, [-, \dots, -]_R, 1)$  is a smooth manifold  $R$  with the structure of a pointed  $n$ -rack such that the  $n$ -ary operation  $[-, \dots, -]_R : R \times R \times \dots \times R \longrightarrow R$  is a smooth mapping. For  $n = 2$ , one recovers Lie racks [1].

**Example 4.2.** Let  $H$  be a Lie group. Then  $H$  endowed with the operation

$$[x_1, \dots, x_n]_G = x_1 x_2 \dots x_{n-1} x_n x_{n-1}^{-1} x_{n-2}^{-1} \dots x_1^{-1},$$

is a Lie  $n$ -rack.

**Example 4.3.** Let  $(H, \{- \dots -\})$  be a group endowed with an antisymmetric  $n$ -ary operation, and  $V$  an  $H$ -module. Define the  $n$ -ary operation  $[-, \dots, -]_R$  on  $R := V \times H$  by

$$[(u_1, A_1), (u_2, A_2) \dots (u_n, A_n)]_R := (\{A_1, \dots, A_n\}u_n, A_1 A_2 \dots A_{n-1} A_n A_{n-1}^{-1} A_{n-2}^{-1} \dots A_1^{-1}).$$

Then  $(R, [- \dots -]_R, (0, 1))$  is a Lie  $n$ -rack.

**Theorem 4.4.** Let  $R$  be a Lie  $n$ -rack and  $g := T_1 R$ . For all  $x_1, x_2, \dots, x_{n-1} \in R$ , the tangent mapping  $\Phi(x_1, x_2, \dots, x_{n-1}) = T_1(\phi(x_1, x_2, \dots, x_{n-1}))$  is an automorphism of  $(\mathfrak{g}, [-, \dots, -]_{\mathfrak{g}})$ .

*Proof.* Since  $\phi(x_1, x_2, \dots, x_{n-1})(1) = [x_1, x_2, \dots, x_{n-1}, 1]_R = 1$ , we apply the tangent functor  $T_1$  to  $\phi(x_1, x_2, \dots, x_{n-1}) : R \longrightarrow R$  and obtain  $\Phi(x_1, x_2, \dots, x_{n-1}) : T_1 R \longrightarrow T_1 R$  which is in  $GL(T_1 R)$  as  $\phi(x_1, x_2, \dots, x_{n-1}) \in {}_n\text{Aut}(R)$  by proposition 2.9. Now by the left distributive property of  $n$ -racks, we have

$$\begin{aligned} & \phi(x_1, x_2, \dots, x_{n-1})(\phi(y_1, y_2, \dots, y_{n-1})(y_n)) = \\ & = \phi(\phi(x_1, \dots, x_{n-1})(y_1), \phi(x_1, \dots, x_{n-1})(y_2), \dots, \phi(x_1, \dots, x_{n-1})(y_{n-1}))(\phi(x_1, \dots, x_{n-1})(y_n)) \end{aligned}$$

which successively differentiated at  $1 \in R$  with respect to  $y_n$ , then  $y_{n-1}$ , until  $y_1$  yields to

$$\begin{aligned} & \Phi(x_1, x_2, \dots, x_{n-1})([Y_1, Y_2, \dots, Y_n]_{\mathfrak{g}}) = \\ & = [\Phi(x_1, x_2, \dots, x_{n-1})(Y_1), \Phi(x_1, x_2, \dots, x_{n-1})(Y_2), \dots, \Phi(x_1, x_2, \dots, x_{n-1})(Y_n)]_{\mathfrak{g}} \quad (4.1) \end{aligned}$$

for all  $Y_1, Y_2, \dots, Y_n \in \mathfrak{g}$ . □

**Theorem 4.5.** *Let  $R$  be a Lie  $n$ -rack and let  $x_1, \dots, x_{n-1} \in R$  corresponding respectively to  $X_1, \dots, X_{n-1} \in \mathfrak{g} := T_1 R$ . Then, the adjoint derivation  $ad\{X_1, \dots, X_{n-1}\} : \mathfrak{g} \longrightarrow gl(\mathfrak{g})$  defined by*

$$ad\{X_1, X_2, \dots, X_{n-1}\}(Y) = [X_1, X_2, \dots, X_{n-1}, Y]_{\mathfrak{g}}$$

*is exactly  $T_1(\Phi)$*

*Proof.* From the proof of theorem 4.4,  $\Phi(x_1, x_2, \dots, x_{n-1}) \in GL(\mathfrak{g})$ . Also, the mapping  $\Phi : R \times R \times \dots \times R \longrightarrow GL(\mathfrak{g})$  satisfies  $\Phi(1, 1, \dots, 1) = I$ , where  $I \in GL(\mathfrak{g})$  is the identity. Differentiating  $\Phi$  at  $(1, 1, \dots, 1)$  yields a mapping  $T_1(\Phi) : T_1(R \times R \times \dots \times R) \longrightarrow gl(\mathfrak{g})$ , where  $gl(\mathfrak{g})$  is the Lie algebra associated to the Lie group  $GL(\mathfrak{g})$ . Also Differentiating the identity (4.1) at  $(1, 1, \dots, 1)$  with respect to  $(x_1, x_2, \dots, x_{n-1})$  yields

$$[X_1, \dots, X_{n-1}, [Y_1, Y_2, \dots, Y_n]_{\mathfrak{g}}]_{\mathfrak{g}} = \sum_{i=1}^n [Y_1, \dots, Y_{i-1}, [X_1, \dots, X_{n-1}, Y_i]_{\mathfrak{g}}, Y_{i+1}, \dots, Y_n]_{\mathfrak{g}}.$$

Hence the mapping  $T_1(\Phi)$  is the adjoint derivation. □

**Corollary 4.6.** *Let  $R$  be a Lie  $n$ -rack and  $\mathfrak{g} := T_1 R$ . Then there exist an  $n$ -linear mapping  $[-, \dots, -]_{\mathfrak{g}} : \mathfrak{g} \times \mathfrak{g} \times \dots \times \mathfrak{g} \longrightarrow \mathfrak{g}$  such that  $(\mathfrak{g}, [-, \dots, -]_{\mathfrak{g}})$  is a Leibniz  $n$ -algebra.*

*Proof.* From the proofs of theorems 4.4 and 4.5, it is clear that the  $n$ -ary operation  $[-, \dots, -]_{\mathfrak{g}}$  is a derivation for itself. □

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